# Type Universes as Allocation Effects

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**Abstract.** In this paper, we explore a connection between type universes and memory allocation. Type universe hierarchies are used in dependent type theories to ensure consistency, by forbidding a type from quantifying over all types. Instead, the types of types (universes) form a hierarchy, and a type can only quantify over types in other universes (with some exceptions), restricting cyclic reasoning in proofs. We present a perspective where universes also describe *where* values are allocated in the heap, and the choice of universe algebra imposes a structure on the heap overall. The resulting type system provides a simple declarative system for reasoning about and restricting memory allocation, without reasoning about reads or writes. We present a theoretical framework for equipping a type system with higher-order references restricted by a universe hierarchy, and conjecture that many existing universe algebras give rise to interesting systems for reasoning about allocation. We present 3 instantiations of this approach to enable reasoning about allocation in the simply typed $\lambda$-calculus: (1) the standard ramified universe hierarchy, which we prove guarantees termination of the language extended with higher-order references by restricting cycles in the heap; (2) an extension with an *impredicative* base universe, which we conjecture enables full-ground references (with terminating computation but cyclic ground data structures); (3) an extension with *universe polymorphism*, which divides the heap into fine-grained regions. (This is a fresh perspectives submission.)

## 1 Introduction

Many type systems have been designed to reason about memory. But what does it mean to reason about memory?

A large amount of work is dedicated to *safety*, ensuring updates, aliasing, *etc.*, do not cause bugs when reading and writing from memory. For example, L3 [1] has a type system that enforces safety with strong updates, *i.e.,* updates that may change the type of already allocated locations on the heap. L3's type system focuses on the safety of reads in the presence of these strong updates, and thus focuses solely on the problem of aliasing. Rust [12] implements a similar system, enabling reasoning about aliasing in the presence of concurrency.

Some type systems enable reasoning about *deallocation*. Region type-and-effect systems [19] were originally designed for static fine-grained memory management, and Rust follows in this tradition.

Much of this work reasons about memory *indirectly*, from how references to memory are used. In Rust, allocation and deallocation is inferred from usage—when a value is created, or copied, it is allocated, and it is deallocated when the value is no longer owned. Regions are also inferred through individual reference reads and writes. Any heap structure, such as the stack of regions structure [19] or stratified regions [4,3], is also inferred through reads and writes. L3 turns out to be terminating, which is enforced indirectly because of the linearity usage requirement on references, but not because of any direct restriction on structure of the heap.

We present a theoretical framework for designing type systems to enable *directly* reasoning about *allocation*, by giving a declarative description of allocation. By declarative, we mean that each type *declares* where values of this type are allocated on the heap, and parameters of the type system *declare* what heap shapes and dependencies are allowed, in contrast to these other type systems where allocation is *inferred* from usage of heap references. For the time being, we do not consider reasoning about aliasing, although we conjecture that existing designs that track aliasing could be integrated into our type systems.

Our framework is based on a *type universe hierarchy*, where the underlying hierarchy can be changed to result in different heap structures. Type universes are used to eliminate inconsistencies in dependent type theories, which can arise when the type of types (the *universe* **Type**), has type **Type**. This, and similar patterns, can lead to inconsistencies by admitting cyclic proofs. A standard solution to this is to stratify universes, each one a member of the next. Instead of considering the **Type** to be in universe **Type**, dependent type theories use a universe hierarchy to assign a level to each universe, so $\mathbf{Type}_i$ is in universe $\mathbf{Type}_{i+1}$, ensuring consistency. In our framework, each type has a universe, but the universe describes where values are allocated, and the universe hierarchy enforces a heap structure for all programs. By changing the universe hierarchy, or level algebra, we impose different heap structures.

We first present an instantiation of our framework with a standard predicative universe hierarchy, which enforces a stratified, acyclic heap. Because the heap is acyclic, we prove this language with higher-order references is terminating. The language is simplified compared to similar work using stratified regions via a type-and-effect system [4,3]. Our language does not require tracking of reads or writes of individual references, nor region inference. We also present the proof of termination, which relies on a semantic notion of garbage collection based on universe level. We conjecture that the semantics might be lead to a declarative syntax for deallocation.

We then abstract this type system to present the main parameters that change the underlying universe algebra to create different heap structures. We give two instantiations of this framework with existing type universe hierarchies, resulting in type systems that enforce different heap structures. In particular, we design a language with a heap with one level where cycles can occur, based on universe hierarchies with a single *impredicative* base universe. We conjecture the language is still terminating and yields full-ground references [15]. We also explore an extension with *universe polymorphism*, which allows a user to name regions of the heap and results in a system with more fine-grained regions and conjecture could be used for static memory management.

In short, we present a fresh perspective on viewing type universes as a system for *simple*, *direct*, *declarative* reasoning about allocation—when and where allocation occurs, and which allocations a computation depends on.

## 2 Type Universes for Acyclic Heaps

We now study how cycles in the heap can be created with higher-order references through a pattern called *Landin's Knot* [10]. We show how to eliminate such cycles in a typed language equipped with a universe type hierarchy. In the type system we present, Landin's Knot is ruled out in a purely declarative way, without any reference usage tracking.

Preventing cycles in the heap is useful to achieve various language design and implementation goals. Reference counting, used to implement memory management, cannot handle (strong) cycles in the heap [13]. Certain cycles through the heap can lead to unrestricted recursion, which one might instead desire to control through only explicit recursive constructs. For example, dependent type theories require strong normalization for decidability of type checking, shown for example by Jutting [9], and termination might be desired to ensure for fairness in concurrent settings [4].

Unfortunately, the common wisdom is that higher-order references introduce cycles in the heap that are difficult to prevent. Landin's Knot, a pattern that uses a function capturing a mutable reference and updates that reference to contain the function itself, is a typical example of these kinds of cycles. This pattern is illustrated through the following diverging program.

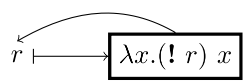
$$
\begin{array}{ll}
id = (\boldsymbol{\lambda}\, x.x) & : \mathbf{Nat} \to \mathbf{Nat} \\
r = \mathbf{new}\; id & : \mathbf{Ref}\; (\mathbf{Nat} \to \mathbf{Nat}) \\
f = (\boldsymbol{\lambda}\, x.(\mathbf{!}\; r)\; x) & : \mathbf{Nat} \to \mathbf{Nat} \\
r := f; & \\
f\; 0 &
\end{array}
$$

Three expressions $id$, $r$, and $f$ are defined in this program. The expression $id$ is the identity function for natural numbers, and $r$ is a reference initialized with $id$. The second function $f$ captures the reference $r$, then expects a natural number $x$ and and applies the function stored in $r$ to $x$. The program diverges because it updates the reference $r$ to store the function $f$, enabling unrestricted recursion in $f$. The function $f$ calls the function stored in $r$, which is now $f$ itself, and the program diverges.

We visualize the heap and dependencies on mutable references to better illustrate the cause of this cycle. In the following diagram, the reference $r$ initially points to a memory location storing the identity function, and the function $f$ depends on $r$. A reference pointing a memory location uses the arrow $\mapsto$, with the location and contents denoted by a box. Code dependent on a reference uses the arrow $\rightarrow$ to distinguish from pointing to memory locations.

$$\boldsymbol{\lambda}\, x.(\mathbf{!}\ r)\ x \longrightarrow r \longmapsto \boxed{\boldsymbol{\lambda}\, x.x}$$

After updating $r$ to $f$, the problematic cycle comes from the combination of the dependency arrow for $f$ on $r$ and $r$ pointing to $f$.

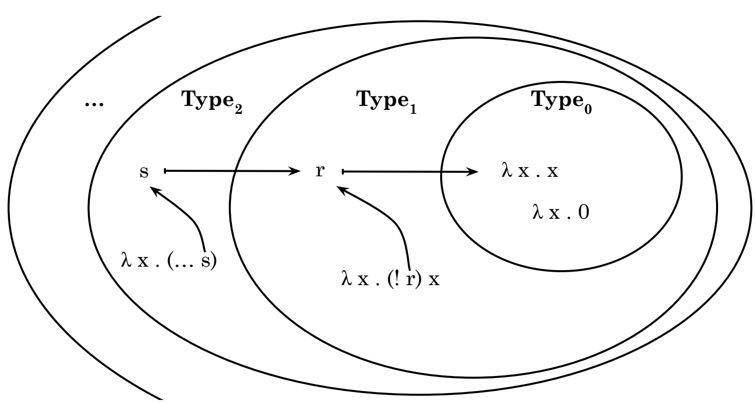



To prevent such cycles, we need to prevent $r$ from containing functions that *depend on* $r$. The first diagram had no cycles, and updates to $r$ with other functions that are not like $f$ has no such cycles: for example, updating $r$ to contain the function $\lambda x.0$.

$$\lambda x.(\mathbf{!}\ r)\ x \longrightarrow r \longmapsto \boxed{\lambda x.0}$$

To use a type universe hierarchy for preventing cycles, we consider $r$ and functions with *any dependencies* on $r$ to be in the same universe. The values that $r$ can store will be at a separate universe, so that $r$ cannot be updated with any value from $r$'s universe, preventing cyclic dependencies. A type universe hierarchy similarly prevents circular reasoning in proofs encoded in dependent type theory, where $\mathbf{Type}_i$ is considered to be of type $\mathbf{Type}_{i+1}$, and not of type $\mathbf{Type}_i$.

To prevent other potential cycles, we also have to prevent $r$ from pointing to any expressions in a *higher* universe than $r$ since such expressions could depend on $r$. This additional restriction imposes an acyclic heap by stratifying the heap such that references can only point "down" in the heap.




This diagram may cause déjà vu for those familiar with models of languages with mutable references, where a similar diagram is also illustrated by Ahmed [2].

The stratification introduced in Ahmed's work is used in the *model* to break the type-world circularity, which we discuss in more detail in Section 2.1. We reflect this stratification into the *syntax* with a type universe hierarchy for a stratified (and thus acyclic) heap. Since functions are also allocated on the heap, our diagram here also includes functions and their dependencies on the heap.

The key kinding rule in our type universe hierarchy for a stratified heap is the rule for determining the universe level of reference types. To require references to point "downward" in the heap, the universe level of a reference type must be one level higher than the type it stores. Here we use :: to indicate kinding a type.

$$\frac{\tau :: \mathbf{Type}_i}{\mathbf{Ref}\ \tau :: \mathbf{Type}_{i+1}}$$

Given a type $\tau$ in universe $\mathbf{Type}_i$, then the type $\mathbf{Ref}\ \tau$ is in universe $\mathbf{Type}_{i+1}$. For example, given *id*'s type is in universe $\mathbf{Type}_0$, then $r$'s type is in universe $\mathbf{Type}_1$.

To complete the design of these stratified higher-order references, we also consider how to determine the universe level of a function type, since functions are also allocated on the heap. The typical predicative function kinding rule in dependent type theories is essentially as follows, where a function's universe level is the maximum of the universe levels of its input and output types.

$$\frac{\tau_1 :: \mathbf{Type}_i \qquad \tau_2 :: \mathbf{Type}_j \qquad k \geq i, j}{\tau_1 \to \tau_2 :: \mathbf{Type}_k}$$

However, since functions are allocated as closures with code *and* environment, the universe information of the environment is lost with this kinding rule. Without considering a function's environment, the functions *id* and $f$ are in the same universe. But having *id* and $f$ in the same universe violates the stratified heap invariant, as $f$'s closure contains $r$. The two functions *id* and $f$ differ as *closures*, since *id* requires no environment, but the function $f$'s environment depends on the reference $r$. This means that the function *id* and $f$, despite having the same input and output types, will be in different universes based on their *environment*. The universe level of a function type is not only influenced by the input and output types, but also the types captured in its environment!

To implement this, we include an annotation on the function arrow. When viewing a function type in isolation, the only information available is the universes of the input and output types, and not the (maximum) universe of the environment. We check that the universe annotation is consistent with the universe of the input and output types. We also separately check the annotation against the environment when the type is used to check a term.

$$\frac{\tau_1 :: \mathbf{Type}_i \qquad \tau_2 :: \mathbf{Type}_j \qquad k \geq i, j}{\tau_1 \xrightarrow{k} \tau_2 :: \mathbf{Type}_k}$$

The types $\mathbf{Nat} \xrightarrow{1} \mathbf{Nat}$ and $\mathbf{Nat} \xrightarrow{0} \mathbf{Nat}$ are both well kinded, but describe different kinds of functions: the former are permitted to capture references in

$$
\begin{array}{llll}
\mathsf{Type} & \tau & ::= & \mathbf{Unit} \mid \mathbf{Nat} \mid \tau \xrightarrow{k} \tau \mid \mathbf{Ref}\ \tau \\
\mathsf{Expr} & e & ::= & x \mid n \mid \langle\rangle \mid \boldsymbol{\lambda}\, x : \tau.e \mid e\ e \mid \mathbf{new}\ e \mid \mathbf{!}\ e \mid e \coloneqq e
\end{array}
$$

**Fig. 1.** $\lambda_{\text{PR}}$ syntax

$\boxed{\Gamma \vdash e : \tau}$

$$
\frac{}{\Gamma \vdash n : \mathbf{Nat}} \qquad \frac{}{\Gamma \vdash \langle\rangle : \mathbf{Unit}} \qquad \frac{x : \tau \in \Gamma}{\Gamma \vdash x : \tau}
$$

$$
\frac{\Gamma, x : \tau_1 \vdash e : \tau_2 \qquad k \geq \textbf{max-level}(\Gamma, \tau_1, \tau_2)}{\Gamma \vdash \boldsymbol{\lambda}\, x : \tau_1.e : \tau_1 \xrightarrow{k} \tau_2} \qquad \frac{\Gamma \vdash e_1 : \tau_1 \xrightarrow{k} \tau_2 \qquad \Gamma \vdash e_2 : \tau_1}{\Gamma \vdash e_1\ e_2 : \tau_2}
$$

$$
\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \mathbf{new}\ e : \mathbf{Ref}\ \tau} \qquad \frac{\Gamma \vdash e : \mathbf{Ref}\ \tau}{\Gamma \vdash \mathbf{!}\ e : \tau} \qquad \frac{\Gamma \vdash e_1 : \mathbf{Ref}\ \tau \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 \coloneqq e_2 : \mathbf{Unit}}
$$

**Fig. 2.** $\lambda_{\text{PR}}$ typing.

universe $\mathbf{Type}_1$, while the latter cannot capture references at all. In our example, $f$'s type would be $\mathbf{Nat} \xrightarrow{1} \mathbf{Nat}$ since it captures the universe $\mathbf{Type}_1$ reference $r$. Then $r$ cannot be updated to $f$, since $f$ is in the same universe $\mathbf{Type}_1$, preventing the cycle.

We now present the full language $\lambda_{\text{PR}}$, where PR stands for *predicative references*, inspired by predicative universe hierarchies. The type system is essentially standard, with a type universe hierarchy to enforce the heap stratification. The type system is syntax directed, as is the kinding system, creating a simple, declarative system for higher-order references with termination without complicated resource tracking.

In Figure 1, we present the syntax of $\lambda_{\text{PR}}$, a simply typed $\lambda$-calculus with higher-order references. The language includes base types, **Nat** and **Unit**, denoted by the metavariable $n$ and expression $\langle\rangle$ respectively. The rest of the syntax is standard, with **new**, **!**, and $\coloneqq$ for initializing, dereferencing, and updating references. The only exception is the annotation on the function type $\tau \xrightarrow{k} \tau$, where $k$ indicates the universe of the function type and can be read as where to allocate a function of this type on the heap.

In Figure 2, we present the typing rules of $\lambda_{\text{PR}}$. The rules are standard, except for the function case. Here we check (or could infer) the annotation on the function type against the current context, $k \geq \textbf{max-level}(\Gamma, \tau_1, \tau_2)$. This side condition requires that the level $k$ of the function type is greater than or equal to the levels of variables in $\Gamma$ and the levels of types $\tau_1$ and $\tau_2$. Note that the context $\Gamma$ could contain more variables (and thus universe levels) than the function actually captures, but through weakening the context one can easily determine the smallest level of a function type by only the variables captured.

$\boxed{\tau :: \textbf{Type}_i}$

$$\frac{}{\textbf{Nat} :: \textbf{Type}_0} \qquad \frac{}{\textbf{Unit} :: \textbf{Type}_0} \qquad \frac{\tau :: \textbf{Type}_i}{\textbf{Ref}\ \tau :: \textbf{Type}_{i+1}}$$

$$\frac{\tau_1 :: \textbf{Type}_i \qquad \tau_2 :: \textbf{Type}_j \qquad k \geq \textbf{max-level}(\tau_1, \tau_2)}{\tau_1 \xrightarrow{k} \tau_2 :: \textbf{Type}_k}$$

**Fig. 3.** $\lambda_{\text{PR}}$ kinding.

The function type annotation relies on kinding the input and output types and the types in $\Gamma$ to determine the maximum universe level. We present the kinding of types in Figure 3. The simple base types **Nat** and **Unit** are of $\textbf{Type}_0$ as expected. The universe level of a function is determined by the annotation, given that this annotation is greater than or equal to the input and output type universe levels. And finally, a reference type is one level higher than the level of the type it stores.

In $\lambda_{\text{PR}}$, our example of Landin's Knot is not well typed, which we demonstrate through type and kind annotations rather than a large derivation tree.

$$\begin{array}{lll}
id = (\boldsymbol{\lambda}\ x.x) & : \textbf{Nat} \xrightarrow{0} \textbf{Nat} & \\
r = \textbf{new}\ id & : \textbf{Ref}\ (\textbf{Nat} \xrightarrow{0} \textbf{Nat}) :: \textbf{Type}_1 & \\
f = (\boldsymbol{\lambda}\ x.(\textbf{!}\ r)\ x) & : \textbf{Nat} \xrightarrow{1} \textbf{Nat} :: \textbf{Type}_1 & \textbf{Type}_1 \text{ due to } r \\
r := f & \text{type error, expected: } \textbf{Nat} \xrightarrow{0} \textbf{Nat} \text{ actual: } \textbf{Nat} \xrightarrow{1} \textbf{Nat} &
\end{array}$$

### 2.1 Proof of Termination

The proof of termination for $\lambda_{\text{PR}}$ uses the standard logical relations technique, an introduction of which can be found in standard textbooks [5]. Our relation models types as sets of normalizing expressions. We prove all well-typed expressions in $\lambda_{\text{PR}}$ are in the set associated with their type in Theorem 1. To be in the set associated with its type, an expression must step to a value, which allows us to conclude that all well typed expressions step to a value, *i.e.,* terminate.

To model a language with mutable references, one has to model the heap, since expressions access and modify the heap. The model of a heap is often referred to as a *world*, and is modelled as a finite map from locations to sets of values modelling types. Since each location is mapped to a set of all possible values, the world represents any possible concrete execution heap.

However, we run into a circularity in reasoning when modelling heaps as worlds, known as the *type-world circularity.*

$$\begin{aligned}
\mathit{Type} &= \mathit{World} \rightarrow \mathit{Set\ of\ Terms} \\
\mathit{World} &= \mathit{Loc} \rightharpoonup \mathit{Type}
\end{aligned}$$

$$\begin{aligned}
\mathcal{V}[\![\mathbf{Ref}\ \tau]\!]_{i+1}(W_{i+1}) &\stackrel{def}{=} \{l \mid W_{i+1}(l) = \mathcal{V}[\![\tau]\!]_i\} \\
\mathcal{V}[\![\tau_i \xrightarrow{k} \sigma_j]\!]_k(W_k) &\stackrel{def}{=} \{\boldsymbol{\lambda}\, x : \tau_i.e \mid \forall W'_k \sqsupseteq_k W_k, v \in \mathcal{V}[\![\tau]\!]_i(\lfloor W'_k \rfloor_i).e[v/x] \in \mathcal{E}[\![\sigma]\!]_j(\lceil W'_k \rceil)\}
\end{aligned}$$

**Fig. 4.** Value relation for reference and function types.

To model a type, we take in a world and produce a set of terms representing that type. The world maps locations to types and must also take in a world, resulting in the following equation with an inconsistent cardinality, that is, the set must contain itself: $\mathit{World} = \mathit{Loc} \rightharpoonup (\mathit{World} \to \mathit{Set\ of\ Terms})$.

*Step-indexed logical relations* were introduced by Ahmed [2] to model languages with mutable references, where the type-world circularity is eliminated using a decreasing number $k$. Using step-indexing, the equations become:

$$\begin{aligned}
\mathit{Type}_{k+1} &= \mathit{World}_k \to \mathit{Set\ of\ Terms} \\
\mathit{World}_k &= \mathit{Loc} \rightharpoonup \mathit{Type}_k
\end{aligned}$$

Unrolling $\mathit{World}_k$, we have the following consistent equation $\mathit{World}_k = \mathit{Loc} \rightharpoonup (\mathit{World}_{k-1} \to \mathit{Set\ of\ Terms}$. In previous work by Ahmed, the decreasing metric $k$ is determined by the number of steps of reduction that an expression takes. Given an expression $e \in \mathit{Type}_{k+1}(W_k)$, the expression $e$ steps "safely", *i.e.,* without type errors, for $k+1$ steps. Ahmed hints that there is a way to stratify types based on syntax, but chooses this alternative semantic approximation in order to model arbitrarily quantified types.

We take the opposite approach to Ahmed and stratify the types based on syntax. The decreasing metric in the type-world equations is instead based on the universe level of a type. The resulting equations are also slightly different, where the universe level matches the world level, and a world level is increased by one based on the universe level of the mapped types.

$$\begin{aligned}
\mathit{Type}_i &= \mathit{World}_i \to \mathit{Set\ of\ Terms} \\
\mathit{World}_{i+1} &= \mathit{Loc} \rightharpoonup \mathit{Type}_i
\end{aligned}$$

Unrolling $\mathit{World}_{i+1}$, we have the following consistent equation $\mathit{World}_{i+1} = \mathit{Loc} \rightharpoonup (\mathit{World}_i \to \mathit{Set\ of\ Terms})$. These equations correspond closely to our kinding rule for references, where reference type $\mathbf{Ref}\ \tau$ is of $\mathbf{Type}_{i+1}$, given $\tau$ is $\mathbf{Type}_i$. The type $\mathbf{Type}_{i+1}$ has access to $\mathit{World}_{i+1}$, that is, locations mapping to $\mathbf{Type}_i$, just as the reference at $\mathbf{Type}_{i+1}$ has access to $\tau$ at $\mathbf{Type}_i$. One may also notice the absence of $\mathit{World}_0$, where locations would be mapped to $\mathbf{Type}_{-1}$, and this corresponds to the idea that values at $\mathbf{Type}_0$ do not need access to the heap. Another interpretation is that these values are valid in any world, because they do not access or update the heap, and thus any world at any level can stand in for $\mathit{World}_0$.

Since the decreasing metric is included in the syntax, the resulting logical relation reads as a simpler version of a step-indexed logical relation, since each expression no longer needs to be paired with a step index $k$. In Figure 4, we

**World extension**
$W'_k \sqsupseteq_k W_k \iff \mathrm{dom}(W_k) \subseteq \mathrm{dom}(W'_k) \wedge \forall l \in W_k . W'_k(l) = W_k(l)$
**Lowering**
$\lfloor W_k \rfloor_i = \{(l, \mathcal{V}[\![\tau]\!]_j) \mid j < i\}$
**Lifting**
$\lceil W_k \rceil = \lambda i. \begin{cases} \lambda l. W_k(l), & 0 < i \leq k \\ \emptyset, & \text{otherwise} \end{cases}$

**Fig. 5.** Additional definitions for worlds.

present the value relation for reference and function types. The resulting sets are simply sets of values. The value relation indexes the *world* at the same level as the universe, avoiding the type-world circularity and ensuring that values are not accessing locations beyond their world level.

The set of values for a reference type **Ref** $\tau$ is indexed by a number $i+1$, since the universe level of **Ref** $\tau$ is always $i+1$ for $\tau$'s level $i$. The world associated with the type **Ref** $\tau$ is also at level $i+1$. Then, the set of values for type **Ref** $\tau$ is all the locations in the current world that map to the set of values associated with $\tau$,*i.e.,* $\mathcal{V}[\![\tau]\!]_i$. The level of worlds associated with such values is necessarily lower than that of the current world $W_{i+1}$, since the set is indexed by $i$.

The set of values for a function type $\tau \xrightarrow{k} \sigma$ relies on additional definitions presented in Figure 5. The first definition is *world extension* $\sqsupseteq_k$, which describes a *future world* $W'_k$ with respect to the world $W_k$. World extension is necessary because a function can be applied later in a future *heap*, and so the model must include function values valid in future *worlds.* The relation guarantees that $W'_k$ has as many locations as $W_k$ at the same types, but may have additional locations allocated.

The next definition *lowers* a world $W_k$ to level $i$. Lowering a world is necessary since the values in the relation $\mathcal{V}[\![\tau]\!]_i$ rely on a world indexed at the same level $i$. However, the available world $W'_k$ is at level $k$, which is potentially incompatible with level $i$. We use the lower operation to remove all locations that map to types with levels *higher* than level $i-1$. The resulting world $\lfloor W'_k \rfloor_i$ is a $World_i$ since there are no locations mapping to types with level greater than $i-1$. This lower operation can be considered a form of semantic garbage collection.

The final definition *lifts* a world $W_k$ to a world that is an intersection of worlds at all possible levels, denoted as $\mathbf{W} = \forall i. World_i$. Such a world $\mathbf{W}$ is used in the expression relation, since an expression may allocate and access levels higher than its universe level $i$. The lift operation can be considered a "cast" from a world with types at universe level $k-1$ allocated to a world that can allocate at higher levels. The current allocated locations in the resulting world are the same as $W_k$, and any other locations at higher levels than $k$ have yet to be allocated, hence denoted as $\emptyset$.

The value relation is defined over a fully annotated function type, where the universe levels of both the input type $\tau$ and output type $\sigma$ are known. Kinding types is easy as shown in Figure 3, and the levels are necessary for indexing the

$$
\begin{aligned}
\mathcal{E}[\![\tau]\!]_i(\mathbf{W}) &\stackrel{def}{=} \{e \mid \forall \mathbf{W}' \sqsupseteq \mathbf{W}, h' \ :\ \mathbf{W}'.\langle h' \mid e\rangle \to^* \langle h'' \mid v\rangle \wedge \\
&\qquad \exists \mathbf{W}'' \sqsupseteq \mathbf{W}'.h''\ :\ \mathbf{W}'' \wedge v \in \mathcal{V}[\![\tau]\!](\lfloor \mathbf{W}'' \rfloor_i)\} \\
\mathcal{G}[\![\cdot]\!](\mathbf{W}) &\stackrel{def}{=} \emptyset \\
\mathcal{G}[\![\Gamma, x : \tau]\!](\mathbf{W}) &\stackrel{def}{=} \{\gamma[x \mapsto v] \mid \tau :: \mathbf{Type}_i \wedge v \in \mathcal{V}[\![\tau]\!]_i(\lfloor \mathbf{W} \rfloor_i) \wedge \gamma \in \mathcal{G}[\![\Gamma]\!](\mathbf{W})\}
\end{aligned}
$$

**Fig. 6.** Expression and context relations.

relations for $\tau$ and $\sigma$ correctly. The set contains functions that, given any value $v$ in the value relation for the input type $\tau$, the body of the function $e$ with the parameter $x$ substituted with $v$ is in the *expression* relation for $\sigma$.

The expression relation defined in Figure 6 describes the set of terminating expressions associated with a type $\tau$ at universe level $i$. At a high level, the set consists of expressions that step to a value $v$ that is in the value relation for $\tau$ at level $i$, that is $\mathcal{V}[\![\tau]\!]_i$. The expression relation is defined over a world $\mathbf{W}$ at any level because during evaluation, an expression may allocate and access levels higher than its universe level $i$, as long as the final value does not depend on them.

There are three worlds $\mathbf{W}$, $\mathbf{W}'$, and $\mathbf{W}''$ in the expression relation. The world $\mathbf{W}$ is considered the model of the minimum or initial concrete heap needed for evaluation. However, expressions can be evaluated in heaps larger than the initial heap, which is why the expression relation includes the future world $\mathbf{W}'$. Then, evaluation is under a concrete heap $h'$ realizing world $\mathbf{W}'$, denoted by $h'\ :\ \mathbf{W}'$. The concrete heap $h'$ contains the same locations as $\mathbf{W}'$, but maps each location to a *single* value from the value relation mapped by $\mathbf{W}'$. There is a final concrete heap at the end of evaluation $h''$, and there must exist a future world $\mathbf{W}'' \sqsupseteq \mathbf{W}'$ related to $h''$. Finally, the value $v$ resulting from evaluation is in the value relation for type $\tau$, with the final world $\mathbf{W}''$ lowered to $i$ since $v$ does not rely on any part of the heap higher than $i$. Lowering also maintains the stratification invariant since values such as functions will be guaranteed not to access the heap at levels greater than $i$, and thus can be "deallocated".

We prove $\lambda_{\text{PR}}$ terminating by proving the fundamental lemma, which relies on a substitution $\gamma$ respecting the context $\Gamma$, as defined in Figure 6.

**Theorem 1.** *If* $\Gamma \vdash e : \tau$ *and* $\tau :: \mathbf{Type}_i$*, then* $\forall \gamma \in \mathcal{G}[\![\Gamma]\!](\mathbf{W}).\gamma(e) \in \mathcal{E}[\![\tau]\!]_i(\mathbf{W})$*.*

This theorem states that if an expression $e$ is well typed at type $\tau$ with universe level $\mathbf{Type}_i$, then $e$ is in the expression relation for $\tau$, *i.e.,* $e$ steps to a value $v$ at type $\tau$. This theorem also applies to open terms by using a substitution $\gamma$.

The key takeaways from this proof is that a syntactic hierarchy of types is able to resolve the type-world circularity by ensuring that there are no cycles in the heap, and such a language is terminating. We need a notion of garbage collection with the lower operation $\lfloor - \rfloor_i$, which potentially allows for a language design where either the garbage collection is explicit in the syntax, or can be inferred by the type system similar to region type-and-effect systems.

## 2.2 Interpretations and Our Framework

We conclude our presentation of $\lambda_{\text{PR}}$ with some intuitive interpretations of the type system, although these connections are not yet formalized. We also present our general framework for instantiating the type system with alternative type universe hierarchies, and present additional instantiations in subsequent sections.

One interpretation of $\lambda_{\text{PR}}$ is as a region type-and-effect system, where each universe level is considered as one very large region. All pure values are in region (level) 0, references to level 0 and functions that close over references are in region (level) 1, and so on. Instead of separate typing and kinding judgments, one could perhaps combine them into one type-and-effect judgment as $\Gamma \vdash e : (\tau, \textbf{region } i)$, where the effect **region** $i$ describes allocation in region $i$ and is determined by kinding $\tau$. Additionally, if a program is made up of values that do not exceed region $n$ (level $n$), then any regions above $n$ can be safely deallocated.

The universe levels can also be interpreted as an allocation effect. Values of types in universe $\textbf{Type}_0$ are allocated at region 0 of the heap. Values of types in universe $\textbf{Type}_1$ are allocated in region 1 of the heap, where values have access to region 0, but not any regions above, and so on for each additional region. Simply based on the type, one knows where a value of a certain type will be allocated on the heap. The structure of the heap is maintained for all programs because values are allocated in their appropriate regions, as enforced by the type hierarchy.

Where a function is allocated is dependent on what parts of the heap the function relies on. Interestingly, a function may allocate and update *new* references and still be considered at $\textbf{Type}_0$, *i.e.,* allocated in region 0, for example:

$$
\begin{aligned}
&ex : \textbf{Nat} \xrightarrow{0} \textbf{Nat} \\
&ex = (\boldsymbol{\lambda}\, x.r = \textbf{new } 3; !r \ + \ x)
\end{aligned}
$$

One can consider the reference $r$ as completely "private" to the function $ex$, and $r$ does not affect the existing structure of the heap nor does $r$ ever leave the function's scope. Under this effect system, functions that do not leak effectful values can be considered "pure" in the sense they do not depend on any part of the heap and thus can be used in any arbitrary heap. Furthermore, the allocations performed inside $ex$ are essentially invisible after the computation ends, and can freely be collected or optimized away.

There may also be an interpretation of $\lambda_{\text{PR}}$ as a coeffect system [16], since the function typing rule is dependent on the context. In coeffect systems, a function type is annotated with an effect derived from the current context, and variables in the context are annotated with their effects. A possible coeffect judgment for function typing is as follows, assuming $\tau$ has level $i$.

$$
\frac{\Gamma, x :^{i} \tau \vdash e : \sigma \qquad k = \textbf{max-effect}(\Gamma)}{\Gamma \vdash \boldsymbol{\lambda}\, x : \tau.e : \tau \xrightarrow{k} \sigma}
$$

For our more general framework, we abstract the design of $\lambda_{\text{PR}}$'s type system into a few "knobs" that can be adjusted to create different designs. The abstraction leads to two parameters that can be adjusted between designs.

1. The highlighted side condition for typing a function, in $\lambda_{\text{PR}}$ this equation is $k \geq \textbf{max-level}(\Gamma, \tau_1, \tau_2)$.
2. The relationship between the universe level of a reference type and the type it stores. In $\lambda_{\text{PR}}$, a reference universe level was always one level higher.

What these equations create is essentially the desired algebra of the universe levels, *i.e.,* the values of the levels and operations over them. In $\lambda_{\text{PR}}$, the algebra consists of natural numbers, a $\geq$ operator, and a successor operator for the level of reference types.

We now explore how to adjust these parameters for alternative designs.

## 3 A Universe Hierarchy with Impredicativity

One type universe hierarchy already in use in dependent type theories allows one universe level to be *impredicative*, and the rest of the levels are predicative [11]. The impredicative level allows propositions to quantify over their own level, *i.e.,* propositions can be sound with cyclic definitions, with some limitations. We conjecture similar reasoning can be used to distinguish between "sound" and "unsound" cycles in the memory heap. Many data structures are encoded using cycles, so we believe there is a correspondence between "sound" cycles for data and "unsound" cycles resulting in nontermination.

We conjecture that *full-ground* references [15], *i.e.,* references to base data and other full-ground references, can be encoded using an impredicative universe level. The idea is that references containing base types will remain at the same level as the base universe $\mathbf{Type}_0$. Functions are not included as base types for full-ground references, and will be at universe $\mathbf{Type}_1$ and above. When a reference stores a function, the reference is no longer full-ground, and is subject to the stratified (predicative) part of the universe hierarchy.

Adding an impredicative universe level is easily done by tweaking our two parameters. For the function case, the equation is changed from $\geq$ to $>$, resulting in a system where pure functions and functions that close over full-ground references are at $\mathbf{Type}_1$. Functions that close over other higher-order references continue up the universe hierarchy similarly to $\lambda_{\text{PR}}$.

$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2 \qquad k > \textbf{max-level}(\Gamma, \tau_1, \tau_2)}{\Gamma \vdash \boldsymbol{\lambda}\, x : \tau_1 . e : \tau_1 \xrightarrow{k} \tau_2}$$

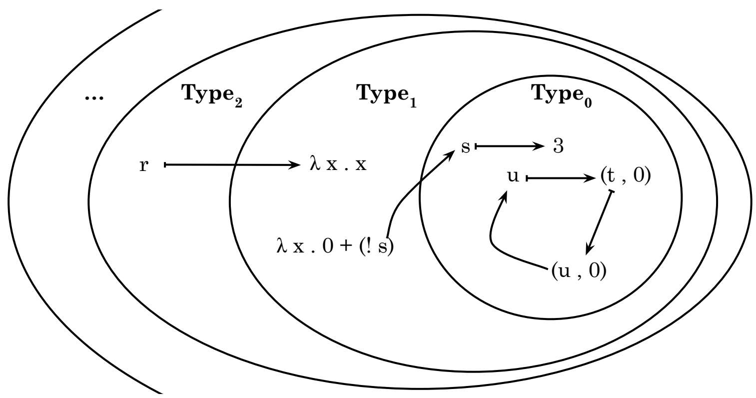
Without this change to $>$, Landin's Knot would be well typed, since both the function *id* and the function that closes over a reference $f$ would be in the same universe.

For reference types, the universe of a reference type now depends on the level of what it stores. When storing a type at base universe $\mathbf{Type}_0$, the reference universe level remains the same. For any types above level 0, the reference universe level is one higher.

$$\frac{\tau :: \mathbf{Type}_0}{\mathbf{Ref}\ \tau :: \mathbf{Type}_0} \qquad \frac{\tau :: \mathbf{Type}_i \qquad i \neq 0}{\mathbf{Ref}\ \tau :: \mathbf{Type}_{i+1}}$$

The resulting type system has full-ground references with sound cycles at $\textbf{Type}_0$, but unsound cycles that result in non-termination are still prevented by the stratification imposed by predicativity. We conjecture that this system is still terminating and has the ability to create cyclic data.

The resulting system will likely have a stratified structured heap as seen in $\lambda_{\text{PR}}$, with cycles occurring in level 0 of the heap.




Here we have a $\textbf{Type}_0$ level of the heap where base data can be allocated along with references to base data, and thus can encode a cyclic list.

## 4 Type Universe Level Polymorphism

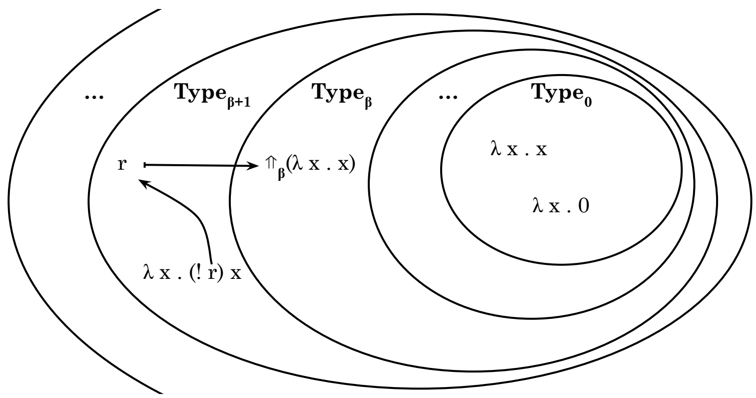
Abstracting explicit levels has been the subject of many research papers on the design of type universe hierarchies. *Universe polymorphism* allows expressions to be polymorphic with respect to universe levels, to eliminate the redundancy of defining the same expressions at different universe levels. Current implementations of dependent type theories use three main techniques for universe polymorphism: explicit quantification for universe levels as done in Agda [18], user constraints as done in Coq [17], and constraint solving based on *typical ambiguity* originally developed by Huet [8] and Harper and Pollack [6] and also implemented in Coq. Recent work by Hou et al. [7] influenced by McBride's *crude-but-effective stratification* [14] has found that an alternative system with an explicit displacement operator is a simple and effective mechanism for universe polymorphism.

We conjecture that universe polymorphism allows us to "name" regions of the heap, resulting in a type system similar to region type-and-effect systems. To implement McBride's *crude-but-effective stratification* version of universe polymorphism, our language needs an explicit displacement operator on both expressions and types. Then, a value typically in one "region" of the heap can be lifted to another region using the displacement operator.

Most base values exist in region 0 (*i.e.,* in universe $\textbf{Type}_0$), and having such a large region that can never be deallocated is poor for fine-grained memory management. We then use the displacement operator to distinguish values into different regions. For example, given a term $e$ of type $\tau$, suppose we would like to allocate $e$ not in region 0 but in some region we call $\beta$. We use the explicit displacement operator $\Uparrow_\beta$ to do this, an operator also defined over types that updates the kind level accordingly.

$$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \Uparrow_\beta e : \Uparrow_\beta \tau} \qquad \frac{\tau :: \mathbf{Type}_\alpha}{\Uparrow_\beta \tau :: \mathbf{Type}_{\alpha+\beta}}$$

Without adjusting the equations $\lambda_{\mathrm{PR}}$, we can now allocate values in different levels of the stratified heap.

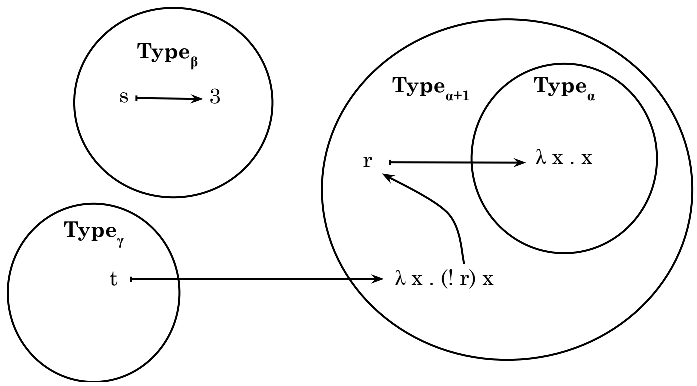



When no expression relies on a particular region (level) $\beta$, any values in and references to level $\beta$ can be safely deallocated. Higher levels are considered separate regions, and allow for fine-grained region-like memory management. The advantages of this system with a lifting operator is that the system is still completely syntax directed, whereas some region type-and-effect systems need to infer the regions based on reads and writes.

However, because we haven't changed the equations from $\lambda_{\mathrm{PR}}$, the resulting regions still have a stratified structure. We conjecture that a more advanced system is possible, where the relationship between a reference universe level and the universe level it stores can be specified through some other relation besides successor.

$$\frac{\tau :: \mathbf{Type}_\alpha \qquad \mathcal{R}(\alpha, \beta)}{\mathbf{Ref}\ \tau :: \mathbf{Type}_\beta}$$

To obtain the stratified variant, $\mathcal{R}$ specifies that $\beta = \alpha+1$. However, to obtain different heap shapes, for example a shape where values allocated in region $\alpha$ are completely separate from references to $\alpha$, we could specify $\alpha \# \beta$ to mean that $\alpha$ and $\beta$ are disjoint. Alternatively, certain values and references to these values could be allocated in the same region by specifying $\alpha = \beta$. With this abstraction, different heap shapes could be formed simultaneously. The pictured heap below has distinct regions $\alpha$, $\beta$, and $\gamma$, where references in $\alpha$ are stratified, references in $\beta$ are in $\beta$, and $\gamma$ exists to contain references separate from values in $\alpha + 1$.




This kinding rule is no longer syntax directed, and may result in a type universe polymorphism system more similar to Coq, where the universes are inferred using constraint solving. Alternatively, a user could declare universes (regions), and constraints between them. For example, the following constraints could be specified to create the heap visualized above, with the assumption that reference types bump the universe level by 1 when no constraints are given.

$$\begin{array}{l}\mathbf{Type}_\alpha \ \# \ \mathbf{Type}_\beta \ \# \ \mathbf{Type}_\gamma \\ \mathbf{Type}_{\beta+1} = \mathbf{Type}_\beta \\ \mathbf{Type}_{\alpha+2} = \mathbf{Type}_\gamma\end{array}$$

These constraints specify that three regions, $\alpha$, $\beta$, and $\gamma$ are all disjoint. However, references to values in region $\beta$ are still in $\beta$, hence the constraint $\mathbf{Type}_{\beta+1} = \mathbf{Type}_\beta$. Additionally, references to level $\alpha+1$, *i.e.,* $\mathbf{Type}_{\alpha+2}$, will be in $\mathbf{Type}_\gamma$, since the reference $t$ is in $\mathbf{Type}_\gamma$.

## 5 Conclusion

What we've studied in this paper is the connection between type universe hierarchies and memory allocation. By studying an example of non-termination using higher-order references, we found a simple mechanism to prevent such cycles by equipping a type system with a type universe hierarchy. The universe hierarchy imposes a structure on the heap for all programs, and we found that the kind of a type describes where values can be allocated on the heap. We are able to distill the essence of this type system to two parameters, one describing the relationship of a function's type level to the levels it captures, and another describing the relationship between a reference type level and what level it stores. These parameters give us a theoretical framework to create type systems that enforce different heap designs by changing the algebra of the universe hierarchy. We believe the simplicity and power of these systems with a universe hierarchy provide a fresh foundation for new language designs for low-level memory reasoning.

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